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Families of Completely Positive Mappings

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The implementation and dilation of families of completely positive mappings on a $*$-algebra are considered.

1. INTRODUCTION

Let $\mathcal{A}$ be a $*$-algebra with an identity I and let ω be a state on $\mathcal{A}$. We may think of $\mathcal{A}$ as an "observable" algebra in which the Hermitian elements represent observables of a quantum mechanical system and ω represents an expectation functional. Unlike the C $*$-algebra framework, the observables in $\mathcal{A}$ can be unbounded. The present framework has certain advantages over that of a C $*$-algebra since the unbounded observables can be treated directly instead of artificially truncating them or taking only bounded functions of them. The $*$-algebra approach is also closely associated with studies in quantum field theory (Borchers, 1962, 1967; Gudder, 1979a). Moreover, there is a widely growing literature on $*$-algebras showing that their structure is almost as rich as that of a C $*$-algebra (Gudder and Hudson, 1978; Inove, 1976, 1977; Lassner, 1972; Lassner and Lassner, 1977; Lassner and Timmermann, 1976; Powers, 1974; Schmüdgen, 1976).

A classical result (Emch, 1976; Powers, 1971) states that a continuous ω-invariant representation of a topological group G on the automorphism group of a C $*$-algebra is implemented by a unitary representation of G on the GNS space. In Section 4 we generalize this result to a $*$-algebra $\mathcal{A}$. In Section 3 we prove that a family of ω-completely positive maps on $\mathcal{A}$ can be implemented by a family of linear operators and a $*$-representation. We also prove that certain families of ω-completely positive maps can be dilated to a semigroup of linear operators. In Section 2 we give the basic definitions and some examples.

2. DEFINITIONS AND EXAMPLES

A map T: $\mathcal{A} \to \mathcal{A}$ is *ω-completely positive* if T is linear and

$$\sum_{i,j=1}^{n} \omega[B_i^* T(A_i^* A_j) B_j] \geqslant 0$$

for any $A_1, \ldots, A_n$, $B_1, \ldots, B_n \in \mathcal{A}$, $n \in N$. For example, if T: $\mathcal{A} \to \mathcal{A}$ is a $*$-homomorphism, then T is ω-completely positive for any state ω. Indeed, then T is linear and

$$\sum_{i,j} \omega[B_i^* T(A_i^* A_j) B_j] = \sum_{i,j} \omega[B_i^* T(A_i)^* T(A_j) B_j]$$

$$= \omega\left[\left(\sum_i T(A_i) B_i\right)^* \sum_j T(A_j) B_j\right] \geqslant 0$$

A family T_s, $s \in S$, of ω-completely positive maps on $\mathcal{A}$ is *unital* if $T_s(I) = I$ for all $s \in S$. Let S be a semigroup with a unit e. If T_s is ω-completely positive for every $s \in S$ and satisfies (1) $T_e = I$, (2) $T_{st} = T_s T_t$ for every $s, t \in S$, then T_s is called a *semigroup of ω-completely positive maps.*

For the case $S = R^+ = \{s \in R: s \geqslant 0\}$, one-parameter semigroups of completely positive maps are used to describe the possibly irreversible dynamics of open quantum mechanical systems (Evans and Lewis, 1977; Kossakowski, 1972; Lindblad, 1976). In a similar way, one-parameter groups $\alpha_t, t \in R$, of automorphisms on $\mathcal{A}$ describe the reversible dynamics of closed quantum mechanical systems. Moreover, a symmetry group for a physical system is given by a representation α: $G \to \operatorname{aut}(\mathcal{A})$ of a group G into the automorphism group $\operatorname{aut}(\mathcal{A})$ of $\mathcal{A}$. Of course, if $T_s \in \operatorname{aut}(\mathcal{A})$, then T_s is unital.

One can also impose continuity conditions on T_s using various topologies. Since we are mainly concerned with the algebraic structure of the framework here, we shall not consider continuity conditions in Section 3. (In Section 4 we shall impose a continuity requirement for symmetry groups.)

We now give an example in which a family of ω-completely positive maps is constructed. In order to avoid certain technicalities we shall assume in the rest of this section that $\mathcal{A}$ is a $C*$-algebra and ω is a faithful state on $\mathcal{A}$. This example can be generalized to a $*$-algebra with an arbitrary state. Let H_ω be the Hilbert space completion of $\mathcal{A}$ relative to the inner product $\langle A, B \rangle = \omega(B^* A)$, and let S be a nonempty set. Suppose we have a $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a collection of bounded linear operators V_s: $H_\omega \to \mathcal{H}$, $s \in S$, satisfying (i) $V_s^* \pi(A) V_s I \in \mathcal{Q}$ for all $s \in S$, $A \in \mathcal{Q}$; (ii) $[V_s^* \pi(A) V_s I]B = V_s^* \pi(A) V_s B$ for all $s \in S$, A, $B \in \mathcal{Q}$. If we define T_s: $\mathcal{Q} \to \mathcal{Q}$ by $T_s(A) = V_s^* \pi(A) V_s I$, then $T_s, s \in S$, is a family of ω-completely positive maps on $\mathcal{Q}$. Indeed, T_s is clearly linear for all $s \in S$, and

$$\sum_{i,j} \omega\left[B_i^* T_s(A_i^* A_j) B_j \right] = \sum_{i,j} \omega\left[B_i^* V_s^* \pi(A_i^* A_j) V_s B_j \right]$$

$$= \sum_{i,j} \langle V_s^* \pi(A_i)^* \pi(A_j) V_s B_j, B_i \rangle$$

$$= \langle \sum_j \pi(A_j) V_s B_j, \sum_i \pi(A_i) V_s B_i \rangle \geqslant 0$$

The family T_s is unital if and only if V_s is an isometry for all $s \in S$. Indeed, if V_s is an isometry, then $T_s(I) = V_s^* V_s I = I$, so T_s is unital. Conversely, if T_s is unital, then

$$V_s^* V_s A = \left[V_s^* \pi(I) V_s I \right] A = T_s(I) A = A$$

for all $A \in \mathcal{Q}$. Since $\mathcal{Q}$ is dense in H_ω, we have $V_s^* V_s x = x$ for all $x \in H_\omega$.

Suppose V_s is an isometry for all $s \in S$ and let P_s: $\mathcal{H} \to \mathcal{H}$ be the projection $P_s = V_s V_s^*$. If $P_s \in \pi(\mathcal{Q})'$, the commutant of $\pi(\mathcal{Q})$, for all $s \in S$, then T_s is a family of $*$-homomorphisms. Indeed, $T_s(A^*) = T_s(A)^*$ since

$$\langle B, T_s(A)^* \rangle = \omega\left[T_s(A) B \right] = \langle T_s(A) B, I \rangle$$

$$= \langle V_s^* \pi(A) V_s B, I \rangle = \langle B, V_s^* \pi(A^*) V_s I \rangle = \langle B, T_s(A^*) \rangle$$

Moreover, T_s is a homomorphism since

$$T_s(AB) = V_s^* \pi(A) \pi(B) V_s I = V_s^* \pi(A) V_s V_s^* \pi(B) V_s I$$

$$= \left[V_s^* \pi(A) V_s I \right] \left(V_s^* \pi(B) V_s I \right) = T_s(A) T_s(B)$$

for all $A, B \in \mathcal{Q}$.

If S is a semigroup with unit e and V_s, π satisfy (i) and (ii′) $[V_{st}^* \pi(A) V_{st} I]B = V_s^* \pi[V_t^* \pi(A) V_t I] V_s B$ for all $s, t \in S$, $A, B \in \mathcal{Q}$; (iii) $V_e^* \pi(A) V_e I = A$ for all $A \in \mathcal{Q}$; then T_s defined above is a semigroup of ω-completely positive maps. Indeed, letting $t = e$ in (ii′) and applying (iii) gives

$$T_s(A) B = \left[V_s^* \pi(A) V_s I \right] B = V_s^* \pi(A) V_s B$$

Hence, (ii) holds and T_s is a family of ω-completely positive maps. Property (1) follows from (iii). To show that Property (2) holds, we have by (ii′)

$$T_{st}(A)=V_{st}^*\pi(A)V_{st}I=V_s^*\pi[V_t^*\pi(A)V_tI]V_sI$$
$$=V_s^*\pi[T_t(A)]V_sI=T_s[T_t(A)]$$

In the next section we shall prove that the converses of the above results essentially hold. In particular, we shall show that every family of ω-completely positive maps on a $*$-algebra has roughly the above form.

3. IMPLEMENTATION

Let $\mathcal{Q}$ be a $*$-algebra with identity I and let ω be a state on $\mathcal{Q}$. The GNS construction (Powers, 1974) provides a unique (to within unitary equivalence) closed $*$-representation π_ω of $\mathcal{Q}$ with domain $D(\pi_\omega)$ in a Hilbert space H_ω and a strongly cyclic vector $x_0\in D(\pi_\omega)$ such that $\omega(A)=\langle\pi_\omega(A)x_0,x_0\rangle$ for all $A\in\mathcal{Q}$. We now prove our main result. The first part of the proof follows Nagy (1955) closely (see also Evans and Lewis, 1977; Gudder, 1979b).

Theorem 1. Let $T_s, s\in S$, be a family of ω-completely positive maps on $\mathcal{Q}$.

(a) There exists a Hilbert space $\mathcal{H}_\omega$, a closed $*$-representation ρ_ω of $\mathcal{Q}$ with domain $D(\rho_\omega)\subseteq\mathcal{H}_\omega$ and a set of linear operators V_s: $D(\pi_\omega)\to\mathcal{H}_\omega$ such that $\pi_\omega[T_s(A)]=V_s^*\rho_\omega(A)V_s$ for all $s\in S$, $A\in\mathcal{Q}$.

(b) If T_s is unital, then V_s is an isometry for all $s\in S$.

(c) If T_s is a unital $*$-homomorphism for all $s\in S$ and P_s: $\mathcal{H}_\omega\to\mathcal{H}_\omega$ is the projection $P_s=V_sV_s^*$, then $P_s\in\rho_\omega(\mathcal{Q})'$. Conversely, if $P_s\in\rho_\omega(\mathcal{Q})'$, then $A\to\pi_\omega[T_s(A)]$ is a $*$-representation.

Proof. (a) Let F be the set of functions f: $\mathcal{Q}\to D(\pi_\omega)$ such that $f(A)=0$ except for finitely many $A\in\mathcal{Q}$ and let $\mathcal{F}=\{\phi\colon \mathcal{Q}\to D(\pi_\omega)\}$. For $\phi\in\mathcal{F}$, $f\in F$, define the sesquilinear form

$$\langle\phi,f\rangle_1=\sum_{A\in\mathcal{Q}}\langle\phi(A),f(A)\rangle$$

For $s\in S$, define $\hat{T}_s$: $F\to\mathcal{F}$ by

$$(\hat{T}_sf)(A)=\sum_{B\in\mathcal{Q}}\pi_\omega[T_s(A^*B)]f(B)$$

It is shown in Gudder (1979b) (for a similar result see Evans and Lewis, 1977; Nagy, 1955) that $\hat{T}_s$ is positive for all $s \in S$; that is, $\langle \hat{T}_s f, f \rangle_1 \geqslant 0$ for every $f \in F$. Define the inner product $\langle \cdot, \cdot \rangle_s$, $s \in S$, on $\hat{T}_s F \subseteq \mathcal{F}$ by $\langle \hat{T}_s f, \hat{T}_s g \rangle_s = \langle \hat{T}_s f, g \rangle_1$ where $f, g \in F$ (it is shown in Evans and Lewis, 1977; Gudder, 1979b; Nagy, 1955 that this is well-defined and is positive definite and hence, an inner product). Let $\mathcal{H}_s, s \in S$, be the Hilbert space completion of $\hat{T}_s F$ relative to $\langle \cdot, \cdot \rangle_s$ and let $\mathcal{H}_\omega = \oplus_{s \in S} \mathcal{H}_s$. For $A \in \mathcal{A}$ and $x \in D(\pi_\omega)$ define $x_A \in F$ by $x_A(B) = x\delta_{A,B}$. Define $V_s(A)$: $D(\pi_\omega) \to \mathcal{H}_\omega$ by $[V_s(A)x]_t = \hat{T}_s x_A \delta_{s,t}$. Then $V_s(A)$ is a linear operator for all $s \in S$, $A \in \mathcal{A}$ and

$$\begin{aligned} \langle V_s(A)x, V_s(B)y \rangle &= \langle \hat{T}_s x_A, \hat{T}_s y_B \rangle_s = \langle \hat{T}_s x_A, y_B \rangle_1 \\ &= \langle \hat{T}_s x_A(B), y \rangle = \langle \pi_\omega[T_s(B^* A)]x, y \rangle \end{aligned}$$

Hence $\pi_\omega[T_s(B^*A)] = V_s(B)^* V_s(A)$ for all $s \in S$, $A, B \in \mathcal{A}$. Define $\rho_\omega(A)$: $\oplus_{s \in S} \hat{T}_s F \to \mathcal{H}_\omega$ by $[\rho_\omega(A)f]_s(B) = f_s(A^* B)$. It is clear that $A \mapsto \rho_\omega(A)$ is an algebra homomorphism. We now show that $\rho_\omega(A)$ leaves $\oplus \hat{T}_s F$ invariant. It suffices to prove that $[\rho_\omega(A)f]_s(B) \in \hat{T}_s F$ for every $f \in \mathcal{H}_\omega$. Since every element of F has the form $g = \Sigma x_{iA_i}$ and

$$(\hat{T}_s \Sigma x_{iA_i})(B) = \Sigma \pi_\omega[T_s(B^* A_i)]x_i$$

we see that

$$\{\pi_\omega[T_s((\cdot)^* A)]x \colon A \in \mathcal{A}, x \in D(\pi_\omega)\}$$

generates $\hat{T}_s F$. Hence, it suffices to prove the above for f of the form $f_s(C) = \pi_\omega[T_s(C^* D)]x$. This follows from

$$\begin{aligned} [\rho_\omega(A)f]_s(B) &= f_s(A^* B) = \pi_\omega[T_s(B^* AD)]x \\ &= (\hat{T}_s x_{AD})(B) \in \hat{T}_s F \end{aligned}$$

We next show that $\rho_\omega(A^*) \subset \rho_\omega(A)^*$. Again, for f defined as above and $g \in \mathcal{H}_\omega$ we have

$$\begin{aligned} \langle g_s, [\rho_\omega(A)f]_s \rangle_s &= \langle g_s, \rho_\omega(A)\pi_\omega[T_s((\cdot)^* D)]x \rangle_s \\ &= \langle g_s, \pi_\omega[T_s((\cdot)^* AD]x \rangle_s = \langle g_s, \hat{T}_s x_{AD} \rangle_s \\ &= \langle g_s, x_{AD} \rangle_1 = \langle g_s(AD), x \rangle = \langle [\rho_\omega(A^*)g]_s(D), x \rangle \\ &= \langle [\rho_\omega(A^*)g]_s, x_D \rangle_1 = \langle [\rho_\omega(A^*)g]_s, \hat{T}_s x_D \rangle_s \\ &= \langle [\rho_\omega(A^*)g]_s, f_s \rangle_s \end{aligned}$$

Hence, ρ_ω is a $*$-representation of $\mathcal{A}$ and has a unique extension to a closed $*$-representation (also denoted by ρ_ω) of $\mathcal{A}$ with domain $D(\rho_\omega)\subseteq \mathcal{H}_\omega$.

For $f\in\mathcal{H}_\omega$, $x\in D(\pi_\omega)$ we have

$$\langle V_s(A^*)f,x\rangle=\langle f,V_s(A)x\rangle=\langle f_s,\hat{T}_s x_A\rangle_s$$
$$=\langle f_s,x_A\rangle_1=\langle f_s(A),x\rangle$$

Hence, $V_s(A)^*f=f_s(A)$. Defining $V_s=V_s(I)$ we have

$$\pi_\omega[T_s(A)]x=\pi_\omega[T_s(AI)]x=V_s(A^*)^*V_s(I)x$$
$$=[V_s(I)x]_s(A^*)=[\rho_\omega(A)V_s(I)x]_s=V_s^*\rho_\omega(A)V_s x$$

(b) If T_s is unital, then $I=\pi_\omega[T_s(I)]=V_s^*V_s$ so V_s is an isometry.

(c) It suffices to show that $\rho_\omega(A)$: $V_sD(\pi_\omega)\to V_sD(\pi_\omega)$. For $x\in D(\pi_\omega)$ we have

$$[\rho_\omega(A)V_sx]_t(B)=(V_sx)_t(A^*B)=(\hat{T}_Ax_I)(A^*B)\delta_{s,t}$$
$$=\delta_{s,t}\pi_\omega[T_s(B^*A)]x=\delta_{s,t}\pi_\omega[T_s(B^*)]\pi_\omega[T_s(A)]x$$
$$=\delta_{s,t}\hat{T}_s[\pi_\omega(T_s(A))x]_I(B)=[V_s\pi_\omega(T_s(A))x]_t(B)$$

Therefore,

$$\rho_\omega(A)V_sx=V_s\pi_\omega[T_s(A)]x\in V_sD(\pi_\omega)$$

The converse is straightforward. ■

We now show that certain families of ω-completely positive maps can be dilated to a semigroup of linear operators. For a related result for C^*-algebras see Evans (1976). A semigroup S is called a *cancellation semigroup* if $st=su$ implies that $t=u$ for any $s,t,u\in S$.

Corollary 2. Let S be a cancellation semigroup with unit e, and let $T_s, s\in S$, be a family of ω-completely positive maps satisfying $T_e=I$. Then there exists a Hilbert space $\mathcal{H}_\omega$ with $H_\omega\subseteq\mathcal{H}_\omega$, a closed $*$-representation ρ_ω of $\mathcal{A}$ with domain $D(\rho_\omega)\subseteq\mathcal{H}_\omega$, and a semigroup of linear operators $W_s, s\in S$, on $\mathcal{H}_\omega$ satisfying

$$\pi_\omega[T_s(A)]=PW_s^*\rho_\omega(A)W_s|D(\pi_\omega) \tag{3.1}$$

for all $s \in S$, $A \in \mathcal{Q}$, where P is the projection from $\mathcal{H}_\omega$ onto H_ω. Moreover, the closed span of the set $\{\rho_\omega(A) W_s x \colon x \in H_\omega, s \in S, A \in \mathcal{Q}\}$ equals $\mathcal{H}_\omega$.

Proof. Define $\mathcal{H}_\omega$, ρ_ω, and V_s as in the proof of Theorem 1. Using the unitary operator $V_e \colon H_\omega \to V_e H_\omega \subseteq \mathcal{H}_\omega$, identify H_ω with $V_e H_\omega$. For $s \in S$, define $W_s \colon \oplus \hat{T}_t F \to \oplus \hat{T}_t F$ as follows. If $f_t = \hat{T}_t x_A$, then $(W_s f)_{st} = \hat{T}_{st} x_A$ and extend by linearity. This is well defined since S is a cancellation semigroup. Clearly, $W_e = I$ and $W_{su} = W_s W_u$ for all $s, u \in S$ so W_s is a semigroup. Now

$$\begin{aligned} \rho_\omega(A) \hat{T}_s x_I(B) = \hat{T}_s x_I(A^* B) = \pi_\omega[T_s(B^* A)]x \\ = \hat{T}_s x_A(B) \end{aligned}$$

and hence, $\rho_\omega(A) \hat{T}_s x_I = \hat{T}_s x_A$. For $x, y \in D(\pi_\omega)$ we then have

$$\begin{aligned} \langle W_s^* \rho_\omega(A) W_s x, y \rangle &= \langle \rho_\omega(A) W_s x, W_s y \rangle \\ &= \langle \rho_\omega(A) W_s \hat{T}_e x_I, W_s \hat{T}_e y_I \rangle = \langle \rho_\omega(A) \hat{T}_s x_I, \hat{T}_s y_I \rangle_s \\ &= \langle \hat{T}_s x_A, \hat{T}_s y_I \rangle_s = \langle \hat{T}_s x_A, y_I \rangle_1 = \langle \hat{T}_s x_A(I), y \rangle \\ &= \langle \pi_\omega[T_s(A)]x, y \rangle \end{aligned}$$

It follows that (3.1) holds. The rest of the proof is straightforward. ∎

4. INVARIANT SYMMETRY GROUPS

Theorem 1 shows that a family T_s of ω-completely positive maps is implemented by a set of linear operators V_s and a closed $*$-representation ρ_ω. An important standard result in C^*-algebras shows that an ω-invariant symmetry group is implemented by a unitary representation (Emch, 1972; Nagy, 1955). In this section we prove that this latter result holds for a $*$-algebra.

Let G be a topological group and let $\alpha \colon G \to \mathrm{aut}(\mathcal{Q})$ be a representation of G into the automorphism group of a $*$-algebra $\mathcal{Q}$. For a state ω on $\mathcal{Q}$ we say that α is *ω-continuous* if $g_i \to g$ (g_i a net in G) implies that $\omega[B\alpha_{g_i}(A)] \to \omega[B\alpha_g(A)]$ for every $A, B \in \mathcal{Q}$. Following the notation of Section 3, π_ω is the GNS representation of $\mathcal{Q}$ with domain $D(\pi_\omega) \subseteq H_\omega$ and strongly cyclic vector $x_0 \in D(\pi_\omega)$.

Theorem 3. Let α: $G \to \mathrm{aut}(\mathcal{A})$ be an ω-continuous representation of the topological group G and suppose ω is α invariant (that is, $\omega[\alpha_g(A)] = \omega(A)$ for all $g \in G$, $A \in \mathcal{A}$). Then there exists a strongly continuous unitary representation U of G on H_ω such that U_g: $D(\pi_\omega) \to D(\pi_\omega)$, $U_g x_0 = x_0$, and

$$\pi_\omega[\alpha_g(A)] = U_g \pi_\omega(A) U_g^* \tag{4.1}$$

for all $g \in G$, $A \in \mathcal{A}$.

Proof. Define $U_g[\pi_\omega(A)x_0] = \pi_\omega[\alpha_g(A)]x_0$, for all $g \in G$, $A \in \mathcal{A}$. This is well defined, since $\pi_\omega(A)x_0 = \pi_\omega(B)x_0$ implies that

$$\begin{aligned} \|\pi_\omega[\alpha_g(A)]x_0 - \pi_\omega[\alpha_g(B)]x_0\|^2 &= \|\pi_\omega[\alpha_g(A-B)]x_0\|^2 \\ &= \omega[\alpha_g((A-B)^*(A-B))] \\ &= \omega[(A-B)^*(A-B)] = \|\pi_\omega(A-B)x_0\|^2 = 0 \end{aligned}$$

Clearly, U_g is a linear operator from $\pi_\omega(\mathcal{A})x_0$ onto itself. Also, U_g is unitary since

$$\begin{aligned} \langle U_g \pi_\omega(A)x_0, U_g \pi_\omega(B)x_0 \rangle &= \langle \pi_\omega[\alpha_g(A)]x_0, \pi_\omega[\alpha_g(B)]x_0 \rangle \\ &= \omega[\alpha_g(B^*A)] = \omega(B^*A) = \langle \pi_\omega(A)x_0, \pi_\omega(B)x_0 \rangle \end{aligned}$$

Since $\pi_\omega(\mathcal{A})x_0$ is dense in H_ω, U_g has a unique unitary extension (also denoted by U_g) to H_ω. Moreover, $U_{gh} = U_g U_h$ for all $g, h \in G$ since

$$\begin{aligned} U_{gh}[\pi_\omega(A)x_0] &= \pi_\omega[\alpha_{gh}(A)]x_0 = \pi_\omega[\alpha_g \alpha_h(A)]x_0 \\ &= U_g[\pi_\omega(\alpha_h(A))]x_0 = U_g U_h[\pi_\omega(A)x_0] \end{aligned}$$

To show that U_g is strongly continuous, suppose that $g_i \to g$. Then

$$\begin{aligned} \|U_{g_i}[\pi_\omega(A)x_0] - U_g[\pi_\omega(A)x_0]\|^2 &= 2\omega(A^*A) - 2\,\mathrm{Re}\,\omega[\alpha_g(A^*)\alpha_{g_i}(A)] \\ &\to 2\omega(A^*A) - 2\,\mathrm{Re}\,\omega[\alpha_g(A^*)\alpha_g(A)] = 0 \end{aligned}$$

A limiting argument then shows that $U_{g_i}x \to U_g x$ for any $x \in H_\omega$.

We now show that

$$U_g \pi_\omega(A) U_g^* \pi_\omega(B)x_0 = \pi_\omega[\alpha_g(A)]\pi_\omega(B)x_0 \tag{4.2}$$

for all $g \in G$, $A, B \in \mathcal{Q}$. Indeed

$$\langle U_g \pi_\omega(A) U_g^* \pi_\omega(B) x_0, \pi_\omega(C) x_0 \rangle = \langle \pi_\omega(A) \pi_\omega[\alpha_{g-1}(B)] x_0, \pi_\omega[\alpha_{g-1}(C)] \rangle$$
$$= \langle \pi_\omega[A \alpha_{g-1}(B)] x_0, \pi_\omega[\alpha_{g-1}(C)] \rangle$$
$$= \omega[\alpha_{g-1}(C^*) A \alpha_{g-1}(B)] = \omega[\alpha_{g-1}(C^* \alpha_g(A) B)]$$
$$= \omega[C^* \alpha_g(A) B] = \langle \pi_\omega[\alpha_g(A)] \pi_\omega(B) x_0, \pi_\omega(C) x_0 \rangle$$

To show that U_g: $D(\pi_\omega) \to D(\pi_\omega)$, let $x \in D(\pi_\omega)$. Since x_0 is strongly cyclic, there exists a net $A_i \in \mathcal{Q}$ such that

$$\pi_\omega(A) \pi_\omega(A_i) x_0 \to \pi_\omega(A) x \tag{4.3}$$

for every $A \in \mathcal{Q}$. Then $\alpha_g(A_i) \in \mathcal{Q}$ and

$$\pi_\omega[\alpha_g(A_i)] x_0 = U_g[\pi_\omega(A_i) x_0] \to U_g x$$

Moreover, $\pi_\omega(A) \pi_\omega[\alpha_g(A_i)] x_0$ is Cauchy since

$$\|\pi_\omega(A) \pi_\omega[\alpha_g(A_i)] x_0 - \pi_\omega(A) \pi_\omega[\alpha_g(A_j)] x_0\|$$
$$= \|\pi_\omega(A) U_g[\pi_\omega(A_i) x_0 - \pi_\omega(A_j) x_0]\|$$
$$= \|U_g \pi_\omega[\alpha_{g-1}(A)][\pi_\omega(A_i) x_0 - \pi_\omega(A_j) x_0]\|$$
$$\leqslant \|\pi_\omega[\alpha_{g-1}(A)][\pi_\omega(A_i) x_0 - \pi_\omega(A_j) x_0]\| \to 0$$

where the second equality follows from (4.2) and the convergence to 0 follows from (4.3). Since all of the $\pi_\omega(A)$ are closed operators, it follows that $U_g x \in D(\pi_\omega)$.

Finally, (4.1) follows from (4.2), the invariance of $D(\pi_\omega)$ under U_g and a limiting argument. ■

It is straightforward to show that the unitary representation U_g is unique up to a unitary equivalence.

REFERENCES

Borchers, H. (1962). "On the structure of the algebra of field operators," *Nuovo Cimento*, **24**, 214–236.

Borchers, H. (1967). "On the theory of local observables," in *Cargese Lectures in Theoretical Physics*, F. Lucat, ed. Gordon and Breach, New York.

Emch, G. (1972). *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*. Wiley-Interscience, New York.

Evans, D. (1976). "Positive linear maps on operator algebras," *Communications in Mathematical Physics*, **48**, 15–22.

Evans, D., and Lewis, J. (1977). "Dilations of irreversible evolutions in algebraic quantum theory." Preprint, Dublin Institute for Advanced Studies, No. 24.

Gudder, S. (1979a). *Stochastic Methods in Quantum Mechanics*. Elsevier-North Holland, New York.

Gudder, S. (1979b). "Positive definite functions" (submitted for publication).

Gudder, S., and Hudson, R. (1978). "A noncommutative probability theory," *Transactions of the American Mathematical Society*.

Inoue, A. (1976, 1977). "On a class of unbounded operator algebras I, II, III," *Pacific Journal of Mathematics*, **65**, **66**, **69**, 77–95, 411–431, 105–115.

Kossakowski, A. (1972). "On quantum statistical mechanics of non-Hamiltonian systems," *Reports on Mathematical Physics*, **4**, 247–273.

Lassner, G. (1972). "Topological algebras of operators," *Reports on Mathematical Physics*, **3**, 279–293.

Lassner, G., and Lassner, G. A. (1977). "Completely positive mappings and unbounded observables," *Reports on Mathematical Physics*, **11**, 113–140.

Lassner, G., and Timmermann, W. (1976). "Classification of domains of operator algebras," *Reports on Mathematical Physics*, **9**, 205–217.

Lindblad, G. (1976). On the generators of quantum dynamical semigroups, *Communications in Mathematical Physics*, **48**, 119–130.

-Nagy, B. Sz. (1955). "Extensions of linear transformations in Hilbert space which extend beyond this space," Appendix to F. Riesz and B. Sz.-Nagy, *Functional Analysis*. Ungar, New York.

Powers, R. (1971). "Self-adjoint algebras of unbounded operators," *Communications in Mathematical Physics*, **21**, 85–124.

Powers, R. (1974). "Self-adjoint algebras of unbounded operators II," *Transactions of the American Mathematical Society*, **187**, 1–33.

Schmüdgen, K. (1976). "Uniform topologies and strong operator topologies on polynomial algebras and on the algebra of CCR," *Reports on Mathematical Physics*, **10**, 369–384.